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Homotopy Approach to Quantum Gravity

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Abstract Gravity may be a quantum-space-time effect. General relativity is quantized by small generic changes in its commutation relations that make its Lie algebras simple on all levels, positing extra variables frozen by self-organization as needed. This quantizes space-time coordinates as well as fields and eliminates physical singularities. Fermi statistics and $\mathrm{sl}(n\mathbb{R})$ Lie algebras are assumed for all levels. Spin $1/2$ is taken to be anomalous, arising from vacuum organization; the spin-statistics relation is incorporated. The gravitational field is quartic in Fermi variables. Einstein's non-commutativity of parallel transport emerges as a vestige of Heisenberg's quantum non-commutativity near the classical limit.

1 Background

Here I simplify general relativity. That is, I make its Lie algebras simple by a small continuous variation or homotopy. This requires vacuum organization analogous to crystallization and ferromagnetism, also suggested by the Higgs field, cosmological inflation.

The simplification strategy quantizes space-time and its deeper structures as well as the fields on it. Several people suggested that space-time might have quantum structure in the early years of the quantum theory. Feynman (ca. 1941) considered the possibility that space-time coordinates were sums of many Dirac spin operators [17]:

$$x^{\mu} = \gamma^{\mu}(1) + \cdots + \gamma^{\mu}(n). \tag{1}$$

Definition 1 A *Feynman space* is a quantum space whose coordinate algebra is a Clifford algebra.

The quantum spaces proposed in this paper are Feynman spaces; their Clifford algebras are Fermi(–Dirac) algebras.

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Snyder (1947) constructed a Lorentz invariant theory with a discrete spectrum for all spacelike coordinates and a continuous one for all timelike coordinates and energy-momentum variables, with a homotopy to the usual classical space-time Lie algebra [41]. No working field theory was built upon Snyder space, mainly for lack of a guiding principle.

Segal (1951) provided one. He pointed out that quantum mechanics and special relativity both result from previous theories by continuous variations that carry singular compound Lie algebras toward generic (structurally stable) ones, perhaps even simple ones, which he showed to be stable. He proposed that one should complete the structural stabilization process that quantization began [38]. Segal may have discarded his idea but it influenced the classic studies of Inönü and Wigner on contraction and the Galilean limit [27], and Gerstenhaber's cohomological theory of Lie algebra stability [25]. These in turn had numerous consequences, many of which I first encountered at this meeting [34]. Others found and used the Segal homotopy independently of Segal [1, 10, 28–30, 32, 35, 42].

Penrose (1971) first quantized a geometric manifold, namely the Euclidean sphere S^2, as opposed to a phase space [36]. He replaces the infinitude of infinite tangent planes of the sphere by a finite Bose combination of spins $1/2$.

This catalyzed other works, including my own reconstruction of Minkowski space-time as a quantum set [19]. That too ignored structural stability and vacuum organization, but has evolved into the present theory.

Flato (1977) used Lie algebra homotopy deeply in deformation quantization independently of Segal and Gerstenhaber [5–7, 24]. Deformation quantization does not consider structural stability or vacuum organization but retains a singular classical manifold that stabilization would eliminate. Simplification and deformation quantization both use homotopy importantly but on diverging paths.

Palev (1977) [35] took a major step toward a stable quantum physics by simplifying the compound algebra of Bose statistics.

Definition 2 A *Palev statistics* is one whose commutation relations are those of a classical Lie algebra and approach the Bose relations in a singular limit.

A Palev statistics can be substituted for Bose statistics everywhere with only small experimental consequences in the present experimental regime.

Vilela-Mendes (1994) found what many had long sought: an atomistic space-time near Minkowski space-time [43]. Inspired by the work of Gerstenhaber, he carried out a structural stabilization of Minkowski space-time and its Heisenberg–Poincaré Lie algebra, positing a fundamental length to set the quantum scale and a suitably large integer to fix the representation.

Definition 3 *Vilela-Mendes space* is a quantum space with preferred coordinates generating a representation of so$(6;\sigma)$ with signature $\sigma = 0$ or 4.

The generalization to other classical Lie groups having the Poincaré group as a singular limit is clear; I will refer to such a quantum spaces as a *generalized Vilela-Mendes space*. It is a Matrix Geometry [16] without its connections and gravity. It is more matrix than the Banks Matrix Model [3] in that its time variable too is a matrix. It combines and unifies the homotopies of Einstein, Heisenberg, de Sitter, and Snyder, who ignored structural stability. Its points could be Palev combinations of simpler spinlike quantum elements.

Baugh (2004) simplified the Poincarë group to a special unitary group SU(n) independently of Palev and Vilela-Mendes. The quantum event of Baugh space can be represented as a pair of Palev sub-events, each with ket space $6\mathbb{C}$ [4].

Shiri-Garakani (2005) simplified a linear dynamics, that of a harmonic oscillator [39, 40]. Structural stability forbids any one-parameter unitary group of time translations, for the Newton commutation relation $[d/dt, t] = 1$ is unstable, and the group U(1) is not simple or stable. At the extremes of system time t, when $t \sim \pm \max |t|$, the multiplicities of the eigenvalues of $|t|$ typically vary rapidly, linearly in $t \mp \max |t|$ and unitarity is a bad approximation, but in the middle times, when $|t| \ll \max \||t|$, unitarity can still be a useful approximation. The usual singular limit keeps only the middle times and so is unitary.

Section 2 formulates a concept of simplification appropriate for a nonlinear field theory. Section 3 simplifies the Einstein space and group of general relativity. Section 4 simplifies the Einstein gravity kinematics and discuss the spin-statistics correlation. Section 5 proposes a simplification of the Einstein-Hilbert dynamics.

2 Simple is Stable

Definition 4 A *simplification* is a homotopy of the structure tensor of a Lie algebra leading to a simple Lie algebra.

Simple Lie algebras are structurally stable (stable, rigid, generic) in that nearby ones are isomorphic.

Simplification is an ill-defined inverse process, like quantization. It goes beyond canonical quantization and has more possible outcomes. The direct process is unsimplification, contraction, flattening, the singular limit. Simplification does not preserve the symmetry group of the flattened structure identically, like renormalization, nor break it completely, like lattice regularization, but varies the group slightly, so that it can remain consistent with past experiments.

Structural stability correlates with dynamical stability. If the Lie algebra elements are to be observables, some compound groups (that is, non-semisimple groups) of present physics force us to infinite-dimensional matrices, in which an energy spectrum can be unbounded below, like those of the classical hydrogen atom and the Dirac one-electron quantum theory. This infinity permits the dynamical instability of unending radiative decay. Any nearby simple group has finite-dimensional representations in which all observables have finite bounded spectra and such dynamical instabilities are impossible.

Almost all quadratic forms are regular, almost all matrices have inverses, almost all determinants are non-zero. Infinities are exceptional, rare; singularity is singular. Experiments, however, have error bars; experiment is generic. Therefore a singular theory is not based entirely on experiment but also postulates some structure of probability 0, usually an idol in the Baconian sense that has become invisible from habituation. This postulation both facilitates the calculations and corrupts them. Infinity in, infinity out. Present physical spaces are built with infinitely many infinite tangent planes and produce infinities. Quantum spaces of simple theories are built instead from finitely many finite quantum elements like spins and produce finite answers.

One calls the singular structure a "contraction" of the regular one [27], although we do not contract a sphere to form a plane, we expand it to infinite radius. People also call the simple structure a "deformation" of the singular one, although here the simple is the norm and the singular is the deformed.

Extra variables are variables that are required before we can simplify most Lie algebras by homotopy, either totally new or replacing existing constants.

For example, in his sole illustration, Segal simplified the Heisenberg commutation relations for one coordinate variable q and momentum p [38]. In his theory, i first stands for a basis element of a real three-dimensional Lie algebra

$$\mathrm{h}(1)\colon \quad qp - pq = i, \qquad iq - qi = 0, \qquad pi - ip = 0. \tag{2}$$

The number $i \in \mathbb{C}$ enters when we represent h(1) in a complex Hilbert space. h(1) simplifies to

$$\mathrm{so}(3)\colon \quad qp - pq = r, \qquad rq - qr = \alpha p, \qquad pr - rp = \beta q. \tag{3}$$

Definition 5 A *simplifier* is an extra variable that permits the Lie algebra to be simplified by a homotopy.

The extra variable in (3) is r.

Simplification strategy requires us to choose:

1. *Simple Lie algebras*. But few simple Lie algebras touch any compound Lie algebra of the present physics. We can try them all.
2. *Vacuum organization* to freeze the simplifiers to constants, like aligned spins in a ferromagnet. One may be able to blame some structural instabilities of present-day physical theory on vacuum organizations posited lately for the Standard Model and cosmology.
3. *A representation of the Lie algebra*. Usually there are infinitely many. For now I build with Fermi algebras, which have unique faithful irreducible representations up to isomorphism. We must still choose how often to iterate Fermi combination. But the dimension grows so explosively with iteration that even a rough count of dimensions can guide this choice.

2.1 Unified Algebras

Simplification unifies the algebras of dynamical variables and space-time coordinates. Heisenberg, emulating Einstein, set out to work solely with observables, and ultimately encoded operations of observation in single-time operators Q of his quantum theory. But his dynamical equations $dQ(t)/dt = [H(t), Q(t)]$ concern not his alleged observables but their histories, observable-valued functions of time $Q(t)$. This theory is singular, like the commutation relation $[d/dt, t] = 1$ of the differential calculus. To evade this singularity I renounce the concept of instantaneous system in favor of system history. Then observables act on simplified Dirac-Schwinger-Feynman probability amplitudes for histories forming a finite-dimensional ket space. The formerly singular "sum over histories" is now merely a finite-dimensional trace.

General covariance can be expressed as invariance under the diffeomorphism group of the space-time manifold, whose Lie algebra is defined by singular relations like $[\partial_\mu, x^\nu] = \delta^\nu_\mu$. This Lie algebra is unstable and must be expressed as singular limit of a stable one. We do so, as prototype for all gauge theories.

Quantum mechanics has one product where classical mechanics had both a product and a Poisson Bracket [26]. Simplification fuses products too. Canonical quantization fused the commutative algebraic product of functions on phase space with the non-associative Poisson bracket product of the same functions. Heisenberg recovered both products by expanding the non-commutative algebra product of quantum mechanics in an $\hbar$ power series.

Simplification merges the inner product $v \cdot w$ of space-time vector fields and the non-associative Lie Bracket $[v, w]_{\mathrm{Lie}}$. Both derive from two associative products of vector fields, as Clifford elements and as differential operators. These merge into one Fermi algebra product of generic relativity. The ket space of the gravitational field is then a multivector space for this Fermi algebra.

Simplification impels us to unify the algebras of field variables and coordinates as well. Hilbert varied gravitational field variables $g_{\mu\nu}(x)$ without varying coordinates $x = (x^\kappa)$. In the resulting Poisson Bracket Lie algebra, $g_{\mu\nu}$ commutes with x^κ. There are no such coordinates in real life. The lattice of rods and clocks imagined by Einstein provides such coordinates at low resolution but would obliterate the system at high resolution. Our actual physical coordinates x^μ are all based on weak signals, usually electromagnetic, that bring us information about the intervening gravitational field as well as the remote event, as in the first astronomical observations of the solar deflection of star images. Such physical coordinates are more relative than general relativity imagined, being relative to the ambient field as well as to the frame of reference. Coordinates in the small that commute with each other and the gravitational field are unnatural in the canonical theory too [8, 9].

2.2 Unified Statistics

Standard q/c field theory is modular in construction, with at least the following stages along the assembly line, defined by ket or sample spaces:

$[F]$: The many-quantum or field operator history $\check{\psi}(x)$.
$[E']$: The single-quantum ket $\psi(x)$.
$[E]$: The event coordinate $x = \int dx$.
$[D]$: The differential dx.

Once coordinates fail to commute, the usual field concept breaks down and needs repair. I represent the simplified field as a combination of events. Its ket space is PV, where V is the ket space for one quantum event and must provide both the field variables and space-time coordinates of the singular limit. A theory is called q/c if it is quantum on the F level and classical on the lower levels $E, D, \ldots$; and so forth. Since the simplified theory is q/q it is not a unified field theory in the original c/c sense of Einstein, but is even more unified, in that it eliminates the distinction between field and space-time as Einstein later advocated.

In a q/c theory, quite different combination processes must connect these levels of assembly. Integration leads from $[D]$ to $[E]$, quantization from $[E]$ to $[E']$, and statistics from $[E']$ to $[F]$. Each level has its own Lie algebras of variables and symmetries to simplify.

The Bohr correspondence principle effectively implies that a homotopy $\hbar \to 0$ connects the dynamical or F levels of q/c theory to c/c theory. To quantize only the dynamical level, however, as Heisenberg and Bohr did, breaks the structural relations to the lower levels. A multilevel quantum theory can simplify and incorporate them, modifying them as needed to improve agreement with experiment.

Definition 6 *Deep simplification* is the extension of the Segal simplification strategy to levels of physics below and including the dynamical.

Where canonical quantization replaces the Poisson Bracket by the commutator of an infinite-dimensional matrix algebra, deep simplification replaces Lie Brackets by commutators of a simple finite-dimensional matrix algebra.

The prototype of multilevel descriptions is a classical algebra S of finite sets. It is a Grassmann algebra over the binary field 2, graded by cardinality, generated by these operations:

Association $\iota : s \mapsto \{s\}$ forms unit sets.

The *product* $s_N \cdots s_1$ of a sequence of N factors represents the serial action of input operations, and the disjoint union of sets, a partial AND operation, with $(\iota s)^2 = 0$. The *sum* $s_N + \cdots + s_1$ represents the parallel action of N terms, and the XOR operation.

$0 \in S$ is the sum of no terms, the non-set; $X = 0$ means that X is not defined. $1 \in S$ is the product of no factors, the empty set; $X = 1$ means that X is empty.

ι is Peano's symbol. Glaserfeld and his school call it unitization since it converts any set into a unit set.

The closest quantum correspondent to the set algebra S is a quantum set theory based on Fermi statistics [20–23]. Its ket space V_S is again a Grassmann algebra, now with coefficient field $K = \mathbb{R}$ or $\mathbb{C}$ instead of 2, graded by cardinality. It is generated by the following operations.

Association is a linear operator $\iota^\dagger : V_S \to V_S$ that maps any vector $v \in S$ of level L into a first-grade creation operator of level $L + 1$:

$$\iota^\dagger v : V_S \to V_S, \quad V_S \ni \psi \mapsto v \vee \psi \in V_S. \tag{4}$$

The set product becomes the Grassmann product, as for Fermi combination. V_S also has an addition operation $v_N + \cdots + v_1$ of quantum superposition. 0 and 1 are as above.

Illustrations: If the u_n are disjoint unit sets, $u_1 + u_2 u_3$ inputs either u_1 (of grade 1) with relative probability $\|u_1\|$ or the union $u_2 u_3$ (of grade 2) with relative probability $\|u_2 u_3\|$. The usual symbol $\{a, b\}$ translates into $\{a\}\{b\}$. In case the ket space V itself consists of products, to generate the Grassmann algebra of V one first associates each element of V. This separates products within braces, if any, from those outside them. Then we form polynomials in the $\iota^\dagger v$ subject to $(\iota^\dagger v)^2 = 0$. The whole process is *Fermi combination* **P**.

The ket space of a Bose combination is $P_+ V$, the symmetric tensor algebra over $\iota^\dagger{}' V$. Let us call such functors from ket spaces of individuals to those of combinations *combinators*. The combinators used here are P for sets, **P** for Fermi, P_+ for Bose, and P_R for Palev combinations with representation R. Any quantum combinator P induces group and algebra finite homomorphisms Π_P from the one to the many. It also induces a Lie algebra homomorphism $\Sigma_P = d\Pi_P$ from the one into the many. If s is an observable Σs is the cumulative sum of s over all the copies of the system in the combination Πs.

In the following V is a vector space with coefficient field $F = \mathbb{R}$ or $\mathbb{C}$, and V^{D} is the dual vector space to V, with typical elements $v \in V$, $u \in V^{\mathsf{D}}$:

Definition 7 [37] $\mathbf{F}V$, the *Fermi algebra* over V, is the Clifford algebra over $V \oplus V^{\mathsf{D}}$ with the quadratic form

$$\|u + v\| = \operatorname{Re} u(v). \tag{5}$$

$\mathbf{F}V$ is the operator algebra of $\mathbf{P}V$, and is generated by Fermi creation and annihilation operators.

Each epoch defines its own stability construct. For example, Segal stabilized Lie algebras against variations in the Lie product but not against departures from the Jacobi identity or co-commutativity. Since these idealizations cause no infinities in the present theory, I retain them. Fermi statistics is not defined by a Lie algebra, however, but by a Clifford algebra, and requires separate consideration. A Clifford algebra is determined by a quadratic form. A quadratic form is structurally stable if and only if it is regular. For Fermi statistics the quadratic form is neutral, hence regular, so Fermi statistics is structurally stable for present purposes.

Canonical quantization uses classical modes of combination on the deeper classical levels and quantum on its quantum surface level. This disrupts interlevel relations; a combination of classical objects cannot be quantum. Deep simplification can preserve interlevel combinatorial relations.

The quantum event of Vilela-Mendes space has a Palev coordinate algebra as though its event is a Palev pair. The Palev Lie algebra is the second-grade part of a Fermi algebra. That is, we may adopt Fermi combination as the quantum correspondent of the power set functor and as the deep statistics. This incorporates the spin-statistics correlation on the dynamical level and only there. Palev events can be formed out of pairs of Fermi events. I assume this is the origin of all bosons (Sects. 2.6, 2.3).

Segal's three variables p, q, r generate the Lie algebra $\mathrm{su}(2) = \mathrm{so}(3)$, in the A, B, and C series. In higher (even) dimensions, however, one must choose between the A series $\mathrm{su}(n)$ and the D series $\mathrm{so}(n)$ and also between the coefficient fields $\mathbb{R}$ and $\mathbb{C}$. I assume this choice of operator algebra must agree with the choice of statistics. The prime candidate is Fermi statistics over the real field, based on these inconclusive indications:

1. Fermions exist.
2. Fermi combination accounts for Bose statistics as well, through Palev statistics.
3. The fundamental representations of the classical groups are Fermi combinations of the "atomic" representations at the terminals of their Dynkin diagrams.
4. Fermi statistics is recursively applied to produce spinor fields, and accords with the empirical spin-statistics correlation (Sect. 2.3).
5. Classical finite set theory, the prototype, is Fermi statistics over the binary field of coefficients 2, recursively applied.
6. Fermi statistics is structurally stable.
7. The Clifford ring of classical gravity is a singular limit of a Fermi algebra.
8. The i of any complex quantum theory destabilizes its Lie algebra.

Assumption 1 All levels have Fermi statistics.

This does not commit us to expressing photons as pairs of neutrinos, as de Broglie proposed. Palev combinations may form out of Fermi elements before the plenum organizes itself into classical space-time and quantum fields. For multilevel Fermi statistics, Level L has the ket space $V[L]$ resulting from $\mathbb{R}$ by L iterations of $\mathbf{P}\iota$'. The ket space for Level $[-1]$ is the empty set $\{\}$ and that for Level $[0]$ is $\{\}\dots$, both trivial as vector spaces, with dimensions -1 and 0. The ket space for Level $[L]$ has $P^L(1)$ dimensions where

$$P^{L+1}(x) = 2^{P^L(x)}, \quad P(0) = 1. \tag{6}$$

Kaluza and Klein imbued space-time with extra internal compact dimensions and created the compactification problem: What compactifies these dimensions? In a fully quantum theory events some coordinates serve as field variables and others as space-time coordinates in the singular limit, all on the same footing in the generic theory. Two events can be combinations linked by a common element, like pinned trusses. They can link along four dimensions of their ket space to form the macroscopic space-time dimensions, leaving all other dimensions comparable to the unit X, like a soap bubble or trussed roof that is many units long in one time direction and two space directions but only one unit thick in the remaining space direction. This replaces the compactification problem by the extension problem: What makes some dimensions grow to macroscopic sizes and not others? As with soap bubbles, this may be a matter of the ambience and the structure and dynamical interaction of the elements. I will not touch this problem here but proceed semi-empirically.

Simplification eliminates non-generic singularities, including the Wronskian singularities of gauge theories and the singularities of propagators on the light cone. Instead of infinite renormalization constants, simplification has finite quantum constants. Vilela-Mendes space has three new homotopy parameters and quantum constants: in the present symbols, a space quantum X, a momentum quantum P, and a large quantum number N, and the usual quantum of angular momentum is $\hbar = \mathsf{XP}$.

The scalar meson in Minkowski space-time and general relativity both have infinite dimensional Lie algebras whose elements depend on arbitrary functions, for example functions of time. Simplification shrinks these Lie algebras to high but finite dimensionality.

2.3 Spins Are Fermi Combinations

A spin $1/2$ is a Fermi combination of creation/annihilation operators of an integer-spin quantum. More algebraically put:

Assertion 1 A spinor space of the orthogonal group $\mathrm{SO}(V_b)$ of some quantum b is the Fermi algebra of V_b, acting on an isotropic vacuum.

Argument The assertion is not new but may have been forgotten. It is the core idea in the classic spinor constructions of Brauer and Weyl, Cartan, and Chevalley [12–14]. Brauer and Weyl liken theirs to "superquantification" by which they mean "second quantization" and Fermi combination in particular. Recall that if W is a quadratic space, any minimal left ideal of the Clifford algebra $\mathbf{Cl}\,W$ is a spinor space for W. Otherwise put, the columns of matrices representing $\mathbf{Cl}\,W$ form a spinor space for W and its orthogonal group $\mathrm{SO}(W)$. The spinors are the columns of the associated Grassmann algebra $\mathbf{P}W$, hence Fermi kets.

This mathematical construction requires physical clarification. As a ket space for some quantum b, W supports the vector representation of $\mathrm{SO}(W)$. In the Minkowski case, by the spin-statistics correlation, a quantum entity b with ket space W would be a boson. In fact the Dirac γ^μ have odd commutation relations (statistics) and even spin. Again, the cited authors form Fermi combinations of b, flouting the spin-statistics relation. The resulting ket space $\mathbf{P}W$, being itself an algebra, has the natural transformation law $\delta_L\Psi = [S(L), \Psi]$ for all $\Psi \in \mathbf{P}W$. But instead of the natural law spinors have the unnatural or *anomalous* transformation law $\delta_L\Psi = S(L)\Psi$. If this spinor construction has physical meaning—and let us suppose that it does if only for the sake of a *reductio*—then it omits some important physical element that absorbs the right action of $S(L)$. As mathematicians we evoke minimal left ideals at will; as physicists we need a physical agent to conserve angular momentum.

The vacuum, the organized ambient plenum, serves. Suppose that a one-dimensional vacuum projector $\Omega = \Omega^2$ is invariant under an antisymmetric generator $S(L)$, $[S(L), \Omega] = 0$. Then $S(L)\Omega = 0$ and

$$\delta_L\Psi\Omega = S(L)\Psi\Omega. \tag{7}$$

That is, the natural spinors are not the multivectors of $\mathbf{P}W$ themselves but the vectors of $\mathbf{P}W{'}\Omega$. Now each part of the spinor construction has a physical counterpart.

1. L relates two experimenters in relative motion (say) assigning kets to some quantum entity b of Level 2 with ket in $W = V_b$.
2. ΣL is the induced action of L on the generic Fermi combination of b's with ket $\Psi \in \mathbf{P}W$.
3. $\Psi\Omega$ applies the combination of creation operations represented by Ψ to a suitable isotropic vacuum Ω.

The spinor space $S = 4\mathbb{R}$ of Minkowski space-time is a square root of the Clifford algebra $C = 2^4 = S \otimes S^{\mathrm{D}}$ of the space-time: $4 = \sqrt{16}$ (not just a square root of space-time as one sometimes hears). The vacuum takes that square root in step 3. In the classic constructions cited, especially Chevalley's, Ω represents a full Dirac sea.

If this standard theory of spinors is to be taken seriously, as I do here, then electron spin 1/2 is as anomalous as anyon spin 1/3. A spinor hides an action on the plenum. Spin is a relative angular momentum of a Fermi combination of several even entities relative to a coherently organized background Ω of many such entities, in the infinite limit.

2.4 Bosonization

It is especially easy to construct excitation quanta with Palev statistics out of a Fermi combination.

Assertion 2 A Fermi combination includes a Palev combination.

Argument The vectors γ_α of the Fermi algebra are Fermi generators. The second-grade tensors $\gamma_\alpha \dagger \gamma_\beta$ in that algebra are $\mathrm{sl}(V)$ generators and therefore Palev generators that have bosonic generators as singular limit. □

The even sub-algebra $C_+ \subset C$ of Fermi statistics can be interpreted as the ket space of a combination with Palev statistics, composed of fermion-pairs γ_{ab}, of even exchange parity.

To actually combine Palev quanta represented by such second-grade operators, one cannot simply multiply the second-grade operators. That would lose the identities of the pairs in the combination. To form a combination from the pair one must first associate them with ι, producing grade-1 units. This is a Lie algebra homomorphism, preserving the commutation relations. The associated pairs still have Palev statistics.

A field history F will be presented as a Fermi combination of events E that are in turn Fermi combinations of *differential events* D:

$$F = \mathbf{P}E - \mathbf{P}^2 D. \tag{8}$$

We encounter three levels $[F] \supset [E] \supset [D]$.

2.5 The Large Number Problem

There seems to be no sign of distinct space-time units in the range of sizes accessible today between subnuclear and cosmic. It seems that a single Fermi combinator $\mathbf{P}$ must bridge from subnuclear to macroscopic ket spaces, just as classically one integration $x = \int dx$ carries us from infinitesimal differentials to cosmological distances. However:

Assertion 3 The number of distinguishable events N_E in the maximal field history F is not greater than the dimensionality of the event ket space V_E:

$$N_E \leq \mathrm{Dim}\, V_E. \tag{9}$$

Argument This follows at once from Assumption 1. □

The successes of the continuum limit suggest that

$$\mathrm{P}^5 1 = 2^{16} \sim 10^5 \ll N_E < \mathrm{Dim}\, V_E < \mathrm{P}^6 1 = 2^{(2^{16})} \sim 10^{(10^{10})}. \tag{10}$$

$\mathrm{Dim}\, V_E$ is so large that I assume that the event E too is composite in the quantum theory as in the classical, of differential elements D: $V_E = \mathbf{P} V_D$. By Assumption 1 of basic Fermi statistics,

$$\mathrm{Dim}\, V_D \sim \log_2 \mathrm{Dim}\, V_E \sim 2^{16}. \tag{11}$$

We can manage with such low dimensionality if we remember that we can apply quantum theory strictly only to a logarithmically small part of the universe; the meta-system including the experimenter requires the rest. For example, even if the number of events in the "history of the universe" were (say) $\sim 10^{20\,000}$, and we could use essentially all of these for the laboratory, the number N_E of events in the maximum feasible quantum field history F that we could control with maximal quantum resolution is no more than $\log_2 N_E \approx 60\,000$, roughly speaking. Then three levels of Fermi analysis suffice to bring us from the event level $[E]$ down to a level $[B]$ of binary elements:

$$V_E \sim \mathbf{P}^3 V_B = 65\,536\,\mathbb{R}. \tag{12}$$

Nevertheless, for a combination of 60 000 Fermi events to be possible, the generic event E must have ket space of at least 60 000 dimensions. Then the differential event D must have kets of at least $\sim$16 dimensions. Since we are used to classical differentials with infinitely many possibilities, 16 is a frighteningly small number, but I will follow this reasoning through. Only 4 of the 16 dimensions develop to macroscopic space-time extension. I use explicitly five nested floating levels $[B] \subset [C] \subset [D] \subset [E]$ and fix them tentatively as the $\mathbf{P}^n \mathbb{R}$ with $2 \le n \le 6$.

2.6 General Spin-Statistics Correlation

For any entity ϵ of the dynamical level F, let $X(\epsilon)$ be the *exchange parity* of ϵ, the operator that exchanges two ϵ kets in a combination $\mathbf{P}\epsilon$. Let $W(\epsilon)$ be the (Wigner) spin parity of ϵ, representing a continuous rotation of an ϵ through 2π. $W(\epsilon)$ has eigenvalue $+1$ for even spin in units of $\hbar/2$ and -1 for odd. The observed spin-statistics correlation is

$$W(\epsilon) = X(\epsilon) \tag{13}$$

for all quanta ϵ of the dynamical level.

Fermi combination converts one ϵ and its vector space of kets into a variable number of ϵ's and a multivector space of kets.

Assertion 4 Assumption 1 (of recursive Fermi statistics) is compatible with the spin-statistics correlation.

Argument Since a spin $1/2$ is an odd combination by Sect. 2.3, it has odd statistics by Assumption 1. Since an integral spin is a combination of an even number of Fermi elements it has even statistics. □

3 Generic Covariance

General relativity replaces Minkowski space and the Poincarǵroup as a model for event space by a collection of spaces with much larger groups:

Definition 8 The *Einstein group* $G_{\mathrm{E}}(\mathcal{M})$ is the group of diffeomorphisms $\mathcal{M} \to \mathcal{M}$ of the manifold $\mathcal{M}$ whose points represent physical events. The *Einstein Lie algebra* $dG_{\mathrm{E}}(\mathcal{M})$ is the Lie algebra of the Einstein group.

$dG_{\mathrm{E}}(\mathcal{M})$ consists of the smooth real vector fields $X = (X^{\mu}(x)\partial_{\mu})$ on $\mathcal{M}$, taken with the Lie product $X_1 \times X_2 := [X_1, X_2]$.

Einstein based general relativity on covariance under the Einstein group; to maintain correspondence I base general quantum relativity on generic covariance, covariance under a simplification $\hat{G}_{\mathrm{E}}$ of the Einstein group $G_{\mathrm{E}}(\mathcal{M})$.

3.1 Vilela-Mendes Space

I use these familiar structures:

Definition 9 The $2N + 1$-dimensional *Heisenberg Lie algebra*

$$\mathrm{h}(N): \quad [p_{\nu}, q^{\mu}] = ir, \qquad [r, p_{\mu}] = 0 = [q^{\mu}, r]. \tag{14}$$

The *Heisenberg algebra* $\mathbf{A}_{\mathrm{H}}(N)$: the algebra of bounded operators providing an irreducible faithful unitary representation $R_{\hbar}\mathrm{H}(N)$ of $\mathrm{h}(N)$ with invariant $r = i\hbar$.

Vilela-Mendes simplified $\mathrm{h}(N)$ to $\mathrm{so}(6;\sigma) =: \mathrm{so}(V)$ of signature σ with generators $o_{\alpha\beta} \in \mathrm{so}(6;\sigma)$ [43]. I index the usual four axes of Minkowski space time with indices $\alpha = 1, 2, 3, 4$ and the two axes of the complex or symplectic plane with indices $\alpha = X, Y$. A convenient complete set of Vilela-Mendes admissible coordinates is

$$\begin{aligned} \hat{x}^{\alpha} &= \mathsf{X} o^{\alpha X}, \\ \hat{p}_{\alpha} &= \mathsf{P} o_{\alpha Y}, \\ \hat{L}_{\alpha\beta} &= \hat{x}_{\alpha}\hat{p}_{\beta} - \hat{x}_{\beta}\hat{p}_{\alpha} = \mathsf{XP} o_{\alpha\beta}, \\ \hat{\imath} &= \mathsf{N}^{-1} o_{XY} \end{aligned} \tag{15}$$

with scale factors X and P having the units of length and momentum. The quantum number N is the maximum eigenvalue of $-io_{XY}$ in a chosen representation R_J of $\mathrm{so}(V)$. The contraction to classical space-time includes the limits

$$\mathsf{X}, \mathsf{P} \to 0, \qquad \mathsf{N} \to \infty \tag{16}$$

and the freezing

$$-io_{XY} \approx l. \tag{17}$$

The classical Minkowski quadratic form is a singular limit of the Casimir operator of the Vilela-Mendes Lie algebra $\mathrm{so}(6;\sigma)$. Designate the Casimir operator therefore by g. To fit Vilela-Mendes space into the Fermi hierarchy I embed $0so(6\mathbb{R};\sigma)$ as a Lie subalgebra within the real linear Lie algebra $\mathrm{sl}(6\mathbb{R})$ of the A series. The generators of the orthogonal group $\mathrm{SO}(V)$ are antisymmetric matrices $o_{\alpha\beta}$. This embedding adduces the traceless parts of the real symmetric matrices $s_{\alpha\beta}$.

Instead of the usual infinite-dimensional representation $R_\hbar$ of the Heisenberg Lie algebra $\mathrm{h}(\mathcal{N})$ by differential operators, one must choose among an infinite number of finite-dimensional representations $R_J\,\mathrm{sl}(6)$ of the linear Lie algebra $0sl(6)$ in $\mathrm{sl}(N)$, labeled by an appropriate collection J of quantum numbers. In what follows, a circumflexed variable is the R_J representative of the un-circumflexed variable. To reduce this choice, I restrict myself provisionally to the (reducible) representations of $0sl(6)$ in the ket space $\mathbf{P}^\kappa 6\mathbb{R}$ that results from κ generations of Fermi combination, starting from the ket space $V = 6\mathbb{R}$ on which the defining representation of $0sl(6)$ acts.

A Vilela-Mendes quantum event can be a pair with Palev statistics constructed by several stages of Fermi combination from a quantum element with ket space $6\mathbb{R}$ of non-Euclidean signature σ.

3.2 Reciprocity

Quantum events like those of Vilela-Mendes space or Baugh space have, in each admissible frame, not only space-time coordinates but also momentum-energy and other coordinates, mixed by the invariance group. This is counterintuitive since it relativizes the construct of absolute space-time point that has pervaded physics since Aristotle. Events are supposed to have space-time coordinates but no momentum-energy coordinates. Geometry may have sprouted from the annually flooded Egyptian fields, where we can suppose that points began as stakes and lines as linen strings, as etymology suggests. These had well-defined momenta, that were small because they and the observer are both well-coupled to each other and to one well-organized condensate, the Earth. A typical Baconian idolization seems to have occurred [2]: Since the momenta were constant they disappeared. In space-time too all actual probes have momentum coordinates that have disappeared. Quantum events restore these energy-momentum variables and a symplectic symmetry between x^μ and p^μ at the event level. Let us call this symplectic $x - p$ symmetry *reciprocity*, though Born's reciprocity acted on a higher level [11]. Since x^μ is local and p^μ, being off-diagonal in x^μ, is non-local, reciprocity breaks locality in the quantum theory. To describe the organization of locality I define a *locality algebra* homomorphic to $2\mathbb{R} \otimes 2\mathbb{R}^{\mathsf{D}}$, the real version of the Pauli spin algebra. It contains the seeds of locality and reciprocity as anticommuting operators.

Definition 10 The *locality algebra* is a 2×2 real matrix subalgebra of the algebra with basis matrices

$$L_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad L_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{18}$$

regarded as defining a quantum space. In a basis of eigenvectors of locality like t and E, the *reciprocity* L_2 generates the transformation

$$L_2\colon \quad \delta x^\mu = p^\mu, \qquad \delta p^\mu = -x^\mu \tag{19}$$

and the *locality* L_3 generates the transformation

$$L_3\colon \quad \delta x^\mu = x^\mu, \qquad \delta p^\mu = -p^\mu. \tag{20}$$

The symmetric operator $L_1 = [L_2, L_3]/2$ generates locality and breaks reciprocity. We can call (18) the *locality basis* of the locality algebra, since the locality is diagonal.

In the Vilela-Mendes so(3, 3; $\mathbb{R}$) Lie algebra, one can reserve the first four basis vectors $\mathbf{1}_A$ $(A = 1, 2, 3, 4)$ for the usual Lorentz so(3; 1; $\mathbb{R}$) Lie algebra and the last two $(A = X, Y)$ for the locality algebra. Since the anti-symmetric operator L_{XY} interchanges x^μ and p^μ, it represents reciprocity L_2; the Vilela-Mendes basis is a locality basis.

3.3 Einstein Group of Feynman Space

For quantization purposes, let us express the general covariance group (diffeomorphism group) algebraically. General relativity, unlike special relativity, implicitly assumes that the coordinates on physical space make up a commutative algebra.

Definition 11 The *general relativity group* or *Einstein group* $\mathrm{E}(\mathcal{N})$ of a classical space-time $\mathcal{N}$ (redefined) is the group of smooth local automorphisms of the commutative coordinate algebra $\mathbf{A}(\mathcal{N})$ of all coordinate functions on $\mathcal{N}$.

Then to simplify the Einstein group it remains only to simplify the coordinate algebra it acts on.

The fully quantum correspondent of that algebra is a full matrix algebra $\mathbf{A}_E$. By Assumption 1 this is the Fermi algebra $\mathbf{A}_E = \mathbf{F}V_D$ over the ket space V_D of the differential event:

Assumption 2 (Generic relativity) The generic event E is a Fermi combination of differential events D.

This makes the event space a Feynman space.

Definition 12 *Generic covariance* is invariance under the simplified Einstein group $G_E = \mathrm{SL}(V_E)$ and its representation on the higher level ket space $V_F = \mathbf{P}V_E$.

G_E must be distinguished from the automorphism group of $\mathrm{sl}(V_E) \subset dG_E$, a much smaller group relevant to special quantum relativity. The ket space $V_E = \mathbf{P}V_D$ of the event has dimensionality $\nu := \mathrm{Dim}\, V_E = 2^{\mathrm{Dim}\, V_D}$. The special quantum coordinates of the event form not an algebra but merely the Lie algebra $\mathrm{sl}(V_E) \subset \mathbf{F}V_D$, embedded in the Fermi algebra by the representation

$$\mathrm{sl}(V_E) \ni \lambda \mapsto \langle\iota|\lambda|\iota\rangle \in \mathbf{F}V_E. \tag{21}$$

The generic quantum coordinates form the algebra $\mathbf{F}V_D$. The generic relativity group G is the group of the regular elements of $\mathbf{F}V_D$, modulo its center $\mathbb{C}$.

This brings the groups of general relativity and quantum theory into alignment as plausible contractions of one generic relativity group $\mathrm{SL}(V_E)$ with event ket space V_E.

The main sequence of contractions or singular limits now proposed is

$$d\hat{G}_E \to \mathrm{sl}((V)C) \to \mathrm{h}(4) \to \mathrm{sl}(2\mathbb{C}). \tag{22}$$

Classical and semiclassical general relativity lie on another line of contractions that dangles from the left-most space of this sequence. This connects the diffeomorphism group of general relativity and the unitary group of quantum theory. Both are plausible contractions of the generic relativity group, along different contraction paths.

By the assumption of Fermi statistics the kets of the physical event and the physical field of generic relativity are spinor spaces of Clifford algebras and at the same time multivector spaces of Fermi algebras. The choice of representation reduces to the choice of how many times the Fermi combination **P** is iterated.

For the Fermi event $E = \mathbf{P}\, D$ with ket space $V = V_E$, the quantum field history $F = \mathbf{P}E$ has ket space $V_F = \mathbf{P}V_E$ and coordinate algebra $\mathrm{A}F = \mathbf{F}V$.

Every Fermi algebra $\mathbf{F}V$ has a Hermitian norm

$$\|x\| := \frac{1}{N}\,\mathrm{Tr}\, x^\dagger x \tag{23}$$

and a quadratic form

$$Q(x) := \frac{1}{N}\,\mathrm{Tr}\, x^2. \tag{24}$$

This quadratic form is indefinite, as is needed for physics, of signature $N(N+1)/2 - N(N-1)/2 = N$, the square root of its dimension. For example, it is a Minkowskian form of signature 2 on the 2×2 matrices.

4 Generic Covariant Kinematics

Einstein used a scalar-valued quadratic form $g(v) := v^{\mu'}(x) g_{\mu'\mu}(x) v^\nu(x)$ on space-time vectors to describe gravity. Next we simplify this kinematics.

Riemann noted that manifolds do not metrize themselves, and supplied a metric "from outside". But Lie groups do; namely, with their Killing form. The generic g could arise not from "outside" as in the Riemann–Einstein theory, but from the natural Killing form of a lower level Lie algebra, as an order parameter of the ambient plenum. Indeed, the Minkowski metrical form $g(p) = g^{\mu\nu} p_\mu p_\nu$ is a singular limit of the Casimir operator K of the ket space $V = 6\mathbb{R}$ underlying Vilela-Mendes space. I infer:

Assumption 3 (Generic gravity) The correspondent in generic gravity of the gravitational form $g(p)$ is the operator

$$\hat{g} := \Sigma^2 K, \tag{25}$$

where K is the Casimir operator

$$K = k_{AB} L^A L^B \tag{26}$$

of the representation in V_F of the Lie algebra $\mathrm{sl}(V_D)$ of Level $[D]$, whose Killing form is k_{AB}.

This combines (3) for some lowest level with (2) for the field level, and excludes (1). To relate this mathematics to the physics of moving bodies and clocks one observes that the Killing form has the defining features of Minkowski's quadratic form: It is invariant under the special covariance group, now simple, and it reduces to proper time $d\tau^2 = dt^2$ in the rest-frame of the singular limit. Before the singular limit there is no rest frame, since the momentum variables do not commute and cannot be set to 0 all together except when all are 0, and in the vacuum one variable o_{YX} is as large as it can be.

This assumes that the generic correspondent of general relativization—the passage from the Poincaré group to the general covariance group—may merely be another combination operation **P**. This fits with the fact that special relativity essentially has one quadratic form while general relativity has an infinite number of tangent spaces each with its quadratic form, all isomorphic but with no natural isomorphism.

In this generic quantum kinematics the gravitational field is not a function on a space-time manifold but an operator on the ket space of a Fermi combination of events. The c space-time manifold is an organization or condensation of the quantum event space.

5 Generic Covariant Dynamics

I may meaningfully assume that the dynamical history multivector is an exponential

$$\Omega = e^{iA/\hbar} \in \mathbf{P}V_E \tag{27}$$

where now A is an action multivector.

We need both an organized and disorganized effective action, each with its own symmetry group. The organized handles processes that do not disorganize the ambient plenum; the other for space-time meltdown and a principled approach to the organization itself. For the organized action we can keep the form of the Hilbert action, which is only general covariant, not generic covariant, resting as it does on several organizations, but replace the operators in it by their generic forms.

I begin with the disorganized action A:

Assumption 4 The action A is generic covariant.

Assertion 5 The action A is a polynomial in the representatives on V_F of the generators of the Lie algebra $\mathrm{so}(V_D)$.

Argument Clear. □

Einstein assumed that the dynamical law was expressed by wave equations of second order in $p \sim \partial_x$. This led Hilbert to an action that is second order in the covariant differentiator $D_\mu(x)$ but not in x, breaking reciprocity and preserving locality. If we imitate Einstein and Hilbert too closely we too will break the reciprocity between x and p, violating generic covariance and arriving at a organized action. We must sacrifice locality for reciprocity.

Following Einstein and Hilbert:

Assumption 5 The action is second-order in the generators of the group $\mathrm{SU}(V_E)$.

Hilbert did not know that his action would eventually be multiplied by the imaginary i to make the phase of a quantum probability-amplitude. For us that i is the frozen form of another generator, o_{YX} in Vilela-Mendes space, and D_{YX} in the generic covariant version. The Einstein-Hilbert imaginary action has differential order two from the organized viewpoint but three from the disorganized viewpoint. We evade this complication for now by assuming that space-time is a Feynman space.

5.1 Space-Time as Feynman Space

Specifically, I use Level $[B]$ to model the Lorentz group, identifying its ket space $V_B = \mathbf{P}^2\mathbb{R}'\Omega$ with the four-dimensional space of real Majorana spinors. To provide enough physical events in V_E requires at least $\kappa = 3$ Fermi combinators: $V_E = \mathbf{P}^3 V_B$. As representation space of $\mathrm{sl}(4\mathbb{R})$, $V_C = 2^{V_B}$ decomposes by grade into $16 = 1 + 4 + 6 + 4 + 1$. When we reduce $\mathrm{sl}(4\mathbb{R}) \to \mathrm{sl}(2\mathbb{C})$, the 6 term reduces according to $6 = 1 + 4 + 1$ where 4 supports the defining representation of the Lorentz group and 1 represents a Lorentz scalar. We can conveniently use this $6 = 1 + 4 + 1$ to represent the $6\mathbb{R}$ underlying Vilela-Mendes space, the 4 representing the usual space-time axes, and $1 + 1$ the XY plane. Later one might use the remaining 10 dimensions of V_C for the defining representation of $\mathrm{sl}(10) \supset \mathrm{so}(10) \supset \mathrm{u}(1) \oplus \mathrm{su}(2) \oplus \mathrm{su}(3)$ incorporating GUT.

$\mathbf{P}^6\mathbb{R}$ has a richer algebra than the Vilela-Mendes space constructed purely with Palev statistics. Hopefully the field ket space $V_F = \mathbf{P}^7\mathbb{R}$ is rich enough to describe both gravity and its Fermi sources, and if so we should use the same $\hat{\imath}$ for both. The V-A weak interaction suggests an imaginary unit of the form

$$\hat{\imath} = \mathsf{N}^{-1}\Sigma^{\kappa}\gamma^{\top}, \quad \kappa \gtrsim 3 \tag{28}$$

whose seed is the top element $\gamma^{\top}$ of its Clifford algebra. Level [2] is the only level whose top element $\gamma^{\top} = \gamma^{4321}$ has the necessary negative square; then $\kappa = 3$ takes us to the event level V_E. I will use γ^{4321} as the seed for $\hat{\imath}$ here.

Then one possible pre-organization action is the Casimir invariant of the simplified Einstein group G_E of the quantum event Level $[E]$ as represented within the group G_F of the quantum field history Level $[F]$. In more detail, let the infinitesimal special linear transformations λ_A be a basis for $\mathrm{sl}(E)$ and $\hat{\lambda}_A = \langle\iota|\lambda_A|\iota\rangle \in \mathbf{A}_F$ be the representation of λ_A in $\mathbf{A}_F$. Then the Killing form is

$$k_{AB} = \mathrm{Tr}\,\Delta\hat{\lambda}_A\Delta\hat{\lambda}_B, \tag{29}$$

and the action is the Casimir invariant K times a quantum constant with the units of action:

$$A = N\hbar K^{AB}\hat{\lambda}_A\hat{\lambda}_B \tag{30}$$

with a possible dimensionless number N [18]. This action and $\hat{\imath}$ commute: they come from seeds on different levels, whose generators anti-commute, and they both have even grade. Each λ is quadratic in the Fermi creation/annihilation operators that generate V_C from V_B. In a sense, (30) is a four-fermion model of the graviton. The combination of four odd quanta into one even occurs below the space-time level, however, unlike four-neutrino proposals.

6 Discussion

The seminal suggestion of Segal [38] that physics evolves toward structural stability by variations in its Lie algebras toward simplicity provides a constructive strategy for theory building: Start from an empirical classical or canonical quantum theory that already works reasonably well in its proper domain, make its Lie algebras simple by the least possible change in their structure tensors, and freeze out the extra variables that this generally requires by positing vacuum self-organization. Perhaps one may think of this as building "from the bottom down," a concept that has influenced this work [31]. Taken seriously, it leads us to quantum events, quantum differentials, and beyond. The photon, the quantum of the electromagnetic field, had to be verified several times to be generally accepted. The quantum event requires at least as much verification, for it attacks even deeper continuity assumptions. The three main gateways to the photon were:

1. Regularization. Planck first introduced the quantum constant h to eliminate the infinite classical heat capacity of electromagnetic cavities and estimated it by fitting the cavity spectral distribution.
2. One-photon observations. Einstein recognized Planck's h as the action quantum of a single photon and estimated it from the photoelectric effect, independently of Planck's estimate. Compton confirmed the photon and estimated h yet again by bouncing photons off free electrons, one at a time.
3. Quantization. Dirac deduced the photon from the canonical commutation relations for the electromagnetic field.

It is harder to isolate one event experimentally than one photon. Space-time is stiff, so its events are strongly coupled, while cavity photons form an ideal gas and are negligibly coupled. For now I used Gateways 1 and 3.

I propose that the macroscopic Clifford ring $C(\mathcal{M})$ of classical gravity on a space-time manifold $\mathcal{M}$ is a singular limit of an underlying Fermi algebra of generic gravity, modified by a vacuum organization. The macroscopic Lie Bracket of classical gravity is a singular limit of the commutator Lie algebra of even-grade elements of this Fermi algebra. The generic simplifications of the gravitational quadratic form and its action could be cumulative forms of Casimir operators of lower-level Lie algebras.

Just as the Einstein group is the automorphism group of the Lie product, the generic Einstein group is the automorphism group of the Lie algebra of the event ket space. This group correspondence is the main tool of this paper, used to construct the generic covariant gravitational kinematics of (25) and dynamics of (30).

To take this extension of relativity seriously we must overcome the enormous apparent difference between space-time coordinates and momenta in our current experience. The difference boils down to the usual assumption that systems can make jumps in p but not in x; or that fields are diagonal in x but not in p. In generic relativity this is a broken $x \leftrightarrow p$ reciprocity symmetry resulting from the organization of locality. To restore the broken reciprocity we must sacrifice locality. Infinitesimal locality is not even defined for generic quantum variables, which have discrete spectra. Quantum theory already permits us to interchange x and p by Fourier transformation. In Vilela-Mendes space, they are interchanged by the operator o_{XY}; in Baugh space by o_{XY} and s_{XY}; in Feynman space by γ_{XY}.

The spectral gap X in x also measures the size of non-local jumps in x. The locality that we currently see reflects a difference in the ranges X and P of the quanta of x and p on the scale of present quantum experiments, and the number of differentials entering into the event.

I pause in the middle of things. Much remains to do: for example, to verify that classical general relativity is indeed a singular limit of this heuristically constructed quantum theory, to simplify the sources of gravity too, to describe the structure of the vacuum organization posited here, and to work out more practical experimental differences from the singular limit.

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